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COMPLETENESS THEOREMS FOR TOPOLOGICAL MODELS

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0. Introduction

This paper is concerned with the language L(Q) which is formed by adding to the first order predicate calculus, L, the quantifier symbol Q. The intended interpretation of $Qx\varphi(x)$ is that the set defined by $\varphi(x)$ is "open".

In recent years there has been increasing interest in both the model theoretic power of generalized quantifiers and the application of logic to other branches of mathematics. In the first direction we have the basic work of Mostowski [8] whose results have been significantly improved by various researchers culminating in Keisler [6]. In his paper Keisler proved a completeness theorem for the "uncountably many" quantifier.

In the other direction we have the fruitful studies of Ax, Kochen, Robinson [9] and others who have successfully applied logic to other areas of mathematics. However, it becomes apparent that the first order predicate calculus is too weak to formulate a theory of topology since the notion of open set appears to involve "higher order" notions. We will attempt by the application of generalized quantifiers to provide a basis for a model theoretic study of topology. At the same time we hope to provide a foundation for further research into the applications of model theory to topology.

In this paper we shall prove a completeness theorem for L(Q) with the following natural set of axioms:

$$Qx(x = x),$$

$$Qx(x \neq x),$$

$$Qx\varphi \wedge Qx\psi \rightarrow Qx(\varphi \wedge \psi),$$

$$\forall y\, Qx\varphi(x, y) \rightarrow Qx\, \exists y\, \varphi(x, y).$$

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We will prove this result in Section 2. Our method uses the Weak Completeness Theorem of [6] using the method of Henkin [3] where the interpretation of Q is some set of subsets of the model. The main difficulty in constructing a topological model is to insure that the interpretation of Q is closed under *arbitrary* unions. The key idea in the proof is the addition of points to every definable non-open set to prevent them from being a union of open sets.

In Section 3 we will apply the main results of Section 2 as well as their proofs to obtain a completeness theorem for $L(Q_{\omega_1}, Q)$ which is L(Q) augmented with the quantifier symbol Q_{ω_1} which has the interpretation "there exists uncountably many". We also prove a completeness theorem for the theory of normal topological models by adding the weaker axiom $\forall x\, Qy(x \neq y)$ to the basic axioms. Other applications include a completeness theorem for $L_{\omega_1,\omega}(Q)$ which is formed by adding Q to $L_{\omega_1,\omega}$.

We conclude this paper by presenting in Section 4 counterexamples to the interpolation and definability problem for L(Q) and a Łoś type theorem for ultraproducts of a topological model. Also we present some remarks on our research into the axiomatization of product topologies and continuous functions. This work is a generalization of the completeness theorem for topological groups announced by the author in [10].

1. Preliminaries

Take the first order predicate calculus L with the identity symbol, $=$. We form the language L(Q), by adding to L a new quantifier symbol Q. Thus L(Q) has three quantifiers $(\exists x)(\forall x)(Qx)$. The set of formulas of L(Q) is the smallest set which contains all the atomic formulas and is closed under $\wedge$, $\sim$, $(\exists x)$, $(\forall x)$ and (Qx). We will use the convention that $\varphi(v_1, \ldots, v_n)$ denotes a formula of L(Q) whose free variables are among $v_1, \ldots, v_n$. Sentences are formulas without free variables.

Take $\mathfrak{A}$ to be a model of L, $q \subseteq \mathscr{S}(A)$, and form $(\mathfrak{A}, q)$. $(\mathfrak{A}, q)$ is called a *weak model* for L(Q). The notion of an n-tuple $a_1, \ldots, a_n \in A$ satisfying a formula $\varphi(v_1, \ldots, v_n)$ of L(Q) in $(\mathfrak{A}, q)$ is defined in the usual manner by induction on the complexity of φ and is denoted by $(\mathfrak{A}, q) \models \varphi[a_1, \ldots, a_n]$. The (Qx) clause is defined as follows: $(\mathfrak{A}, q) \models (Qv_m)\varphi[a_1, \ldots, v_m, \ldots, a_n]$ if and only if $\{b \in A \mid (\mathfrak{A}, q) \models \varphi[a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n]\} \in q$ where $\varphi(v_1, \ldots, v_n)$ is a formula of L(Q) and $m \leq n$. The other clauses in the definition are the familiar ones for L. It is easy to check by induction on the complexity of φ that if all the free variables of $\varphi(v_1, \ldots, v_n)$ are among $v_1, \ldots, v_n$ and if $a_1 = b_1, \ldots, a_n = b_n$ then: $(\mathfrak{A}, q) \models \varphi[a_1, \ldots, a_n]$ if and only if $(\mathfrak{A}, q) \models \varphi[b_1, \ldots, b_n]$.

The axioms for L(Q) are

(i) $\forall x\, (\varphi \leftrightarrow \psi) \rightarrow (Qx\varphi \leftrightarrow Qx\psi)$,

(ii) $Qx\varphi(x) \leftrightarrow Qy\varphi(y)$.

The rules of inference for L(Q) are the same as for L, namely:

Modus Ponens: from φ, $\varphi \rightarrow \psi$ infer ψ,
Generalization: from φ infer $(\forall x)\varphi$.

A more explicit presentation of L(Q) as well as the proofs of the following theorems are found in Keisler [6].

Theorem 1.1 (Weak Completeness Theorem). *Σ is consistent in* L(Q) *if and only if Σ has a weak model* $(\mathfrak{A}, \mathrm{q})$, *where the elements of* q *are all* L(Q) *definable.*

Let $L_{\omega_1,\omega}$ be the infinitary logic with countable conjunctions and finitary quantification. Then $L_{\omega_1,\omega}(Q)$ is the logic formed by adding to $L_{\omega_1,\omega}$ the quantifier symbol Q.

More formally the axioms and rules of inference for $L_{\omega_1,\omega}(Q)$ are just those for L(Q) and $L_{\omega_1,\omega}$. For a more detailed exposition and the proof of the following theorem see [6].

Theorem 1.2 (Completeness Theorem for $L_{\omega_1,\omega}(Q)$). *A sentence φ of* $L_{\omega_1,\omega}(Q)$ *is consistent if and only if φ has a weak model.*

Now we proceed to give several more definitions and theorems which will be needed in this paper.

Definition 1.3 (Tarski and Vaught). $(\mathfrak{B}, r)$ is said to be an *elementary extension* of $(\mathfrak{B}, r)$, in symbols $(\mathfrak{A}, \mathrm{q}) \prec (\mathfrak{B}, r)$, if and only if $A \subseteq B$ and for all formulas $\varphi(x_1, \ldots, x_n)$ of L(Q) and all $a_1, \ldots, a_n \in A$ we have $(\mathfrak{A}, \mathrm{q}) \models \varphi[a_1, \ldots, a_n]$ iff $(\mathcal{L}, r) \models \varphi[a_1, \ldots, a_n]$. A sequence $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha < \gamma$ of weak models is said to be an *elementary chain* if and only if we have $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \prec (\mathfrak{A}_\beta, \mathrm{q}_\beta)$ for all $\alpha < \beta < \gamma$.

The union of an elementary chain $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha < \gamma$ is the weak model $(\mathfrak{A}, \mathrm{q}) = \bigcup_{\alpha<\gamma} (\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$ such that $\mathfrak{A} = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ and

$$\mathrm{q} = \{S \subset A \mid \text{for some } \beta < \gamma,\ \beta \leq \alpha < \gamma \text{ implies } S \cap A_\alpha \in \mathrm{q}_\alpha\}$$

These definitions enable us to state

Theorem 1.4 *Let* $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha < \gamma$ *be an elementary chain and let* $(\mathfrak{A}, \mathrm{q})$ *be the union. Then for all* $\alpha < \gamma$

$$(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \prec (\mathfrak{A}, q).$$

We now present the last model theoretic theorem needed for weak models.

Theorem 1.5 (Löwenheim–Skolem Theorem). (a) *Let* $(\mathfrak{A}, \mathrm{q})$ *be a weak model of* L(Q) *and* $\aleph$ *cardinal such that* $|\mathrm{L}| \leq \aleph \leq |A|$. *Then there is a weak model* $(\mathfrak{B}, r)$ *such that* $(\mathfrak{B}, r) \prec (\mathfrak{A}, \mathrm{q})$ *and* $|B| = \aleph$.

(b) *Let* $(\mathfrak{A}, \mathrm{q})$ *be a weak model of* L(Q) *and* $\aleph$ *a cardinal such that* $|\mathrm{L}| + |A| \leq \aleph$. *Then there is a weak model* $(\mathfrak{B}, r)$ *such that* $(\mathfrak{A}, \mathrm{q}) \prec (\mathfrak{B}, r)$ *and* $|B| = \aleph$.

In order to study the first order theory of topology we will now give the basic definitions.

Definition 1.6. Let X be a set and $\tau \subseteq \mathscr{P}(X)$. Then the pair (X, τ) is called a topology if (1) X and $\emptyset$ are in τ, (2) the intersection of any two elements of τ is an element of τ and (3) the union of any collection of elements of τ is an element of τ. We call the elements of τ the "open" sets.

Let $(\mathfrak{A}, q)$ be a weak model of L(Q). We call $(\mathfrak{A}, q)$ *topological* if q is a topology on A. Now we can proceed to prove the basic theorems about first order topology.

2. The basic completeness theorem

We shall now introduce a simple set of axioms for topology. The following natural set of axioms are easily seen to be true in every topological model

(A0) All axiom schemes for L(Q),

(A1) $Qx(x = x)$,

(A2) $Qx(x \neq x)$,

(A3) $Qx\varphi \wedge Qx\psi \rightarrow Qx(\varphi \wedge \psi)$,

(A4) $\forall y\, Qx\varphi(x, y) \rightarrow Qx\, \exists y\, \varphi(x, y)$.

We will now write L(Q) for the logic formed by (A0)–(A4) above. It is called the topological quantifier logic.

Using these axioms we can prove the following completeness theorem for topological models. Let Σ be a set of sentences of L(Q). Then Σ has a topological model if and only if Σ is consistent in L(Q).

Before we proceed to prove the theorem we will present the trivial topological fact which forms the basis for the method of proof:

Let (X, τ) be a topological space. Then if $Y \subseteq X$ and Y is not open then there is a $c_y \in Y$ such that if $\mathcal{O} \in \tau$ and $\mathcal{O} \subseteq Y$ then $c_y \notin \mathcal{O}$. In other words any non-open set has at least one point which is not in any open subset of it. With this we shall proceed to prove the completeness theorem for topology.

Theorem 2.1. *Let T be an* L(Q) *theory. Then T is consistent in* L(Q) *if and only if T has a topological model.*

Proof. (if direction). Suppose T has a topological model, $(\mathfrak{A}, q)$, then T is consistent in L(Q) since (A0)–(A4) are all true in every topological model.

(only if direction). Assume T is consistent in L(Q). Then we need to construct a topological model.

For each formula, $\varphi(x, y_1, \ldots, y_n)$, of L(Q) add a new function symbol $f^\varphi(y_1, \ldots, y_n)$ and given any formula, $\psi(x, z_1, \ldots, z_m)$ let ψ^φ be

$$\forall y_1, \ldots, y_m, z_1, \ldots, z_m (Qx\psi(x, z_1, \ldots, z_m) \wedge \sim Qx\varphi(x, y_1, \ldots, y_n) \wedge$$
$$\wedge \forall x (\psi(x, z_1, \ldots, z_m) \to \varphi(x, y_1, \ldots, y_n)) \to$$
$$\to \varphi(f^\varphi(y_1, \ldots, y_n), y_1, \ldots, y_n) \wedge \sim \psi(f^\varphi(y_1, \ldots, y_n), z_1, \ldots, z_n)).$$

That is if φ defines a non open set then f^φ picks out a point from φ which is not in any open subset of φ definable from ψ. Let $L' = \cup \{f^\varphi \mid \varphi \in L(Q)\}$.

We will now show that $T' = T \cup \{\psi^\varphi \mid \varphi, \psi$ are formulas of $L(Q)\}$ is consistent in L'(Q). To do this we only need to show that every finite subset is consistent.

Let $\psi_i^{\varphi_i} 0 \leq i \leq m$ be a finite set of formulas as above. Since T is consistent in L(Q) and by the weak completeness theorem we get a weak model $(\mathfrak{A}, q)$ of T where q is generated by the L(Q) definable sets. For each $0 \leq i \leq m$, $a_1, \ldots, a_{j_i} \in A$ such that $(\mathfrak{A}, q) \models \sim Qx\varphi_i(x, a_1, \ldots, a_{j_i})$ let $f^{\varphi_i}(a_1, \ldots, a_{j_i})$ be some element of

$$[\varphi_i(x, a_1, \ldots, a_{j_i})]^{(\mathfrak{A},q)} - \bigcup_{i=0}^{m} [\exists z_1, \ldots, z_{m_i}(Qx\psi_i(x, z_1, \ldots, z_{m_i}) \wedge \psi_i(x, z_1, \ldots, z_{m_i}))]^{(\mathfrak{A},q)}.$$

That is $f^{\varphi_i}(a_1, \ldots, a_{j_i})$ is some element of $[\varphi_i(x, a_1, \ldots, a_{j_i})]^{(\mathfrak{A},q)}$ which is not in any open set defined by some $\psi_i(x, z_1, \ldots, z_{m_i})$. This is possible since T is consistent with (A4). Otherwise let $f^{\varphi_i}(a_1, \ldots, a_{j_i})$ be any element of A. Then $(\mathfrak{A}, f^{\varphi_1}, \ldots, f^{\varphi_m}, q) \models T \cup \{\psi_i^{\varphi_i} \mid 0 \leq i \leq m\}$. Thus T' is consistent.

By another application of the weak completeness theorem T' has a weak model $(\mathfrak{B}, r')$ where r' is generated by L'(Q) definable sets. Let r be the restriction of r' to the L(Q) definable sets. By (A3), r forms a basis for a topology which we will call r^*.

We will now prove that $(\mathfrak{B}, r) \equiv (\mathfrak{B}, r^*)$ which will show that T has a topological model. We prove this by induction on the complexity of the formula (of L(Q)) with parameters from B. The only difficult case is for $Qx\varphi(x, b_1, \ldots, b_n)$. We need to show that $(\mathfrak{B}, r) \models Qx\varphi(x)$ if and only if $(\mathfrak{B}, r^*) \models Qx\varphi(x)$. If $(\mathfrak{B}, r) \models Qx\varphi(x)$ then $(\mathfrak{B}, r^*) \models Qx\varphi(x)$ since $r \subseteq r^*$. So suppose $(\mathfrak{B}, r^*) \models Qx\varphi(x)$ but that $(\mathfrak{B}, r) \models \sim Qx\varphi(x)$. Then $[\varphi(x, b_1, \ldots, b_n)]^{(\mathfrak{B},r)} = \bigcup_{\alpha \in I} [\varphi_\alpha(x)]^{(\mathfrak{B},r)}$ where $(\mathfrak{B}, r) \models Qx\varphi_\alpha(x)$ for each $\alpha \in I$. But note that

$$f^\varphi(b_1, \ldots, b_n) \in [\varphi(x, b_1, \ldots, b_n)]^{(\mathfrak{B},r)} - \bigcup_{\alpha \in I} [\varphi_\alpha(x)]^{(\mathfrak{B},r)}.$$

So we obtain a contradiction which shows that $(\mathfrak{B}, r) \equiv (\mathfrak{B}, r^*)$. Thus T has a topological model.

Note that by the Löwenheim–Skolem Theorem $(\mathfrak{B}, r^*)$ can have any cardinality $\geq |L|$.

Corollary 2.2. *The set of sentences valid in every topological model is recursively enumerable in the language.*

Proof. Theorem 2.1 shows that a sentence is provable if and only if it is valid so we are done.

Corollary 2.3 (Compactness Theorem). *Let T be an* L(Q) *theory. Then T has a topological model if and only if every finite subset of T has a topological model.*

Proof. An easy application of the basic completeness theorem.

We can now state and prove a Löwenheim–Skolem Theorem for topological models using some of the ideas from Theorem 2.1.

Theorem 2.4. (a) *Let $(\mathfrak{A}, q)$ be a topological model. Then for any $\aleph \geq |L| + |A|$ there is a topological model $(\mathfrak{B}, r)$ such that $(\mathfrak{A}, q) \prec (\mathfrak{B}, r)$, $|B| = \aleph$.*

(b) *Let $(\mathfrak{A}, q)$ be a topological model for* L(Q). *Then for any $|L| \leq \aleph \leq |A|$ there is a topological model $(\mathfrak{B}, r) \prec (\mathfrak{A}, q)$ such that $|B| = \aleph$.*

Proof. (a) By the remark at the end of Theorem 2.1 we can find a $(\mathfrak{B}, r) \succ (\mathfrak{A}', q)$ where $|B| = \aleph$ and r is a topology. Let r^* be the topology generated by the definable open sets. One can easily show that $(\mathfrak{B}, r^*) \succ (\mathfrak{A}', q)$.

(b) Let f^φ, φ a formula of L(Q), be as in Theorem 2.1, $\mathfrak{A}' = (\mathfrak{A}, f^\varphi)_{\varphi \in L(Q)}$. Then $(\mathfrak{A}', q)$ has an elementary submodel $(\mathfrak{B}, r)$, $|B| = \aleph$, by the Löwenheim–Skolem Theorem for weak models. Again, if r^* is the topology generated by the L(Q) definable elements (with parameters) of r we have the theorem.

Note that by the above we have constructed a model, $(\mathfrak{B}, r^*)$, such that $\mathcal{K}((\mathcal{L}, r^*)) \leq |\mathfrak{B}|$ where $k((\mathfrak{B}, r^*)) = \inf\{|\beta| : \beta \text{ is a basis for } r^*\}$.

3. Applications of the completeness theorem

We shall present various applications of the basic completeness theorem.

In 3.1 we show that the theory of normal topological models is equivalent to the L(Q) theory generated by the axiom $\forall x\, \mathrm{Q}y(x \neq y)$ which is a logical formulation of the fréchet axiom in topology. If this theorem is applied to a topological model with a countable basis we obtain a metric space model.

In 3.2 we prove a completeness theorem for $L(Q_{\omega_1}, Q)$ using the completeness theorem of Section 2 and the completeness theorem for $L(Q_{\omega_1})$ proved in Keisler [6]. As an application of the completeness theorem we axiomatize first countable, second countable, and separable topological models. Another application includes transfer theorems to $L(Q_{\omega_\alpha}, Q)$.

In 3.3 we conclude the applications by proving a completeness theorem for $L_{\omega_1\omega}(Q)$ which is the logic formed from $L_{\omega_1\omega}$ by the addition of (Qx) and new axioms for topology.

3.1

Before we state and prove the main theorem we will need several definitions from topology. For further background one should refer to Cullen [2].

Definition 3.1.1. A topological space is called *fréchet* if each point is closed.

Definition 3.1.2. A topological space is called *regular* if each point and disjoint closed set can be separated by disjoint open sets. (We assume that points are closed).

Definition 3.1.3. A topological space is called *normal* if every pair of disjoint closed sets can be separated by disjoint open sets. (We assume that points are closed).

Definition 3.1.4. A topological space is called *0-dimensional* if its topology is generated by sets which are both open and closed (clopen).

Now we can proceed to show that the L(Q) theory of fréchet topological models is the same as the L(Q) theory of 0-dimensional normal topological models.

One should note that the fréchet topological property is expressible by the axiom $\forall x\, Qy(x \neq y)$. First we need to prove the following important lemma.

Lemma 3.1.5. *If T is consistent with $\forall x\, Qy(x \neq y)$ in* L(Q) *then, if $Qx \sim \psi(x)$, $Qx \sim \varphi(x)$, and $\sim \exists x\,(\psi(x) \wedge \varphi(x))$ are consistent with T (i.e. ψ and φ define disjoint closed sets) then $\forall x\,(\psi(x) \rightarrow U^{\psi,\varphi}(x))$, $\forall x\,(\varphi(x) \rightarrow \sim U^{\psi,\varphi}(x))$, $QxU^{\psi,\varphi}(x)$, and $Qx \sim U^{\psi,\varphi}(x)$ are consistent with T where $U^{\psi,\varphi}$ is a new one place predicate symbol. The conclusion means that $U^{\psi,\varphi}$ and $\sim U^{\psi,\varphi}$ define open sets which separate ψ and φ.*

Proof. We need only show the lemma for countable T then using the compactness theorem for L(Q) we obtain it for all T. Thus let $(\mathfrak{A}, q) \models T$ be a countable topological model. Now we need the following trivial fact from topology: If (X, τ) is a fréchet space then for any open set $\mathcal{O}$ and any non-open subset V of $\mathcal{O}$ we have that $\mathcal{O} - V$ is infinite. This is easy to see since in a fréchet space the complement of a finite set is open which would mean that V is open.

Take $\{\delta_i(x)\}_{i\in\omega}$ to be an enumeration of the formulas of L(Q) with one free variable and parameters from A such that

$$(\mathfrak{A}, q) \models \sim Qx\delta_i(x) \quad \text{for each} \quad i.$$

Also let $\{\sigma_k(x)\}_{k\in\omega}$ be an enumeration of the formulas of L(Q) with one free variable and parameters from A such that

$$(\mathfrak{A}, q) \models Qx\sigma_k(x) \quad \text{for every} \quad k.$$

Let h be a bijection from $\mathbf{N}$ onto $\mathbf{N}\times\mathbf{N}$ and if $h(k)=\langle k_1,k_2\rangle$ define $(h(k))_1=k_1$ and $(h(k))_2=k_2$.

Henceforth, we will assume that $[\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})}$ is not an open set since then there would already be a separation

Take $x_0\neq y_0$ to be points such that

$$x_0, y_0\in[\sigma_{(h(0))_1}(x)]^{(\mathfrak{A},\mathrm{q})}\cap(A-([\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})}\cup[\delta_{(h(0))_2}(x)]^{(\mathfrak{A},\mathrm{q})})),$$

if possible. Otherwise let $x_0, y_0\in(A-[\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})})$ which is always possible since $(A-[\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})})$ is infinite because ψ and φ are not open.

Proceeding, we define $x_{n+1}\neq y_{n+1}\notin\{x_1,y_1,\ldots,x_n,y_n\}$ to be such that $x_{n+1},y_{n+1}\in[\sigma_{(h(n+1))_1}(x)]^{(\mathfrak{A},\mathrm{q})}\cap(A-([\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})}\cup[\delta_{(h(n+1))_2}(x)]^{(\mathfrak{A},\mathrm{q})}))$ if possible. Otherwise let $x_{n+1},y_{n+1}\in(A-[\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})})$ which is possible because it is infinite. The reason for such a torturous definition will become apparent shortly.

We claim that $U=\{x_i\}_{i\in\omega}\cup[\psi]^{(\mathfrak{A},\mathrm{q})}$ and $A-U$ separate $[\psi]^{(\mathfrak{A},\mathrm{q})}$ and $[\varphi]^{(\mathfrak{A},\mathrm{q})}$. This is easy since the sequences $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ were picked to miss both $[\psi]^{(\mathfrak{A},\mathrm{q})}$ and $[\varphi]^{(\mathfrak{A},\mathrm{q})}$ and to be disjoint.

Now let q^* be the topology generated by $\mathrm{q}\cup\{U, A-U\}$. We will show that $(\mathfrak{A},\mathrm{q})\equiv(\mathfrak{A},\mathrm{q}^*)$. The only difficult case is the Q clause.

Since $\mathrm{q}\subseteq\mathrm{q}^*$ it is easy to see that if

$$(\mathfrak{A},\mathrm{q})\models Qx\chi(x)\quad\text{then}\quad(\mathfrak{A},\mathrm{q}^*)\models Qx\chi(x).$$

Suppose

$$(\mathfrak{A},\mathrm{q}^*)\models Qx\chi(x)\quad\text{and}\quad(\mathfrak{A},\mathrm{q})\models\sim Qx\chi(x).$$

Then $[\chi'(x)]^{(\mathfrak{A},\mathrm{q})}=[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}=\mathcal{O}_1\cup(\mathcal{O}_2\cap U)$, $\mathcal{O}_1\cup(\mathcal{O}_2\cap(A-U))$ or $\mathcal{O}_1\cup(\mathcal{O}_2\cap U)\cup(\mathcal{O}_3\cap(A-U))$ where $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3\in\mathrm{q}$ since all other cases are trivial.

We will show only the first case since the other cases are analogous. We know that $[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}=\mathcal{O}_1\cup(\mathcal{O}_2\cap U)$ and that $\mathcal{O}_2=\bigcup_{k\in I}[\theta_k(x)]^{(\mathfrak{A},\mathrm{q})}$ where the $\theta_k(x)$ are formulas of L(Q) with parameters in A which define open sets. We can assume for each $k\in I$ $[\theta_k(x)]^{(\mathfrak{A},\mathrm{q})}\cap(A-([\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})}\cup[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}))$ is finite since otherwise $\mathcal{O}_2\cap U$ would not be a subset of $[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}$. Hence, because the topology is fréchet and $[\theta_k(x)]^{(\mathfrak{A},\mathrm{q})}$ and $(A-[\psi\vee\varphi]^{(\mathfrak{A},\mathrm{q})})$ are open sets we have an open set $\mathcal{O}$ such that $(\mathcal{O}_2\cap U)\subseteq\mathcal{O}\subseteq[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}$. This means that $\mathcal{O}_1\cup(\mathcal{O}_2\cap U)\subseteq\mathcal{O}_1\cup\mathcal{O}\neq[\delta_m(x)]^{(\mathfrak{A},\mathrm{q})}$. This is a contradiction and thus the lemma is proved.

We now have sufficient machinery to prove the main theorem of this section.

Theorem 3.1.6. *Let T be an* L(Q) *theory and κ a regular infinite cardinal. Then T is consistent in* L(Q) *with* $\forall\ Qy(x\neq y)$ *if and only if T has a 0-dimensional normal topological model of cardinality κ.*

Proof. (if direction). Easy since normal implies fréchet.

(only if direction). Step 1. Let $(\mathfrak{A},\mathrm{q})$ be a topological model of T of cardinality κ. By applying Lemma 3.1.5 and Theorem 1.4 (union of elementary chains) κ times we obtain a regular, 0-dimensional topological model $(\mathfrak{A}^*, q^*)$ where q^* is generated by the definable open sets (with parameters).

Since $(\mathfrak{A}^*, q^*)$ is topological and regular we can expand it by adding new functions and a binary predicate from a new language, L^*, such that $(\mathfrak{A}^*, f^{\varphi}_{\varphi \in L^*(Q)}, U^{\mathfrak{A}^*}(x, y), q^*)$ models T, $\forall x\, Qy U(x, y)$, $\forall x\, Qy \sim U(x, y)$, for each ψ, $\varphi \in L^*(Q)$, $\forall y\, (Qx\psi(x) \wedge \psi(y) \rightarrow \exists z\, (U(z, y) \wedge \forall x\, (U(z, x) \rightarrow \psi(x))))$ and ψ^{φ}. (i.e the ψ^{φ}'s and the f^{φ}'s are as in the completeness theorem of Section 2 and guarantee that we can generate a topology from the weak model. Also $U(x, y)$ defines a collection of clopen sets which insure that the topology is 0-dimensional and regular.)

We will now define an elementary chain of $L^*(Q)$ topological models, $(\mathfrak{B}_\beta, r_\beta)$, $\beta < \kappa$, as follows:

If $\alpha = 0$ then $(\mathfrak{B}_0, r_0) = (\mathfrak{A}^*, f^{\varphi}_{\varphi \in L^*(Q)}, U^{\mathfrak{A}^*}(x, y), q^*)$.

If $\alpha = \beta + 1$ then we define a theory T_α to be:

$$T_\alpha = \begin{cases} \mathcal{T}\mathcal{H}((\mathfrak{B}_\beta, r_\beta)) \\ \sim \varphi(c_\varphi, b_1, \ldots, b_k) \text{ where } (\mathfrak{B}_\beta, r_\beta) \models \sim Qx\varphi(x, b_1, \ldots, b_k) \\ \psi(c_\varphi) \text{ where } f^{\varphi}(b_1, \ldots, b_k) \in [\psi(x)]^{(\mathfrak{B}_\beta, r_\beta)} \text{ and} \\ \qquad (\mathfrak{B}_\beta, r_\beta) \models Qx\psi(x). \end{cases}$$

T_α is consistent since if $f^{\varphi}(b_1, \ldots, b_k) \in [\psi_i(x)]^{(\mathfrak{B}_\beta, r_\beta)}$ for $0 \leq i \leq m$ then $f^{\varphi}(b_1, \ldots, b_k) \in \bigcap_{0 \leq i \leq m} [\psi_i(x)]^{(\mathfrak{B}_\beta, r_\beta)}$. Hence $\bigcap_{0 \leq i \leq m} [\psi_i(x)]^{(\mathfrak{B}_\beta, r_\beta)} \not\subseteq [\varphi(x, b_1, \ldots, b_k)]^{(\mathfrak{B}_\beta, r_\beta)}$. This is because a finite intersection of open sets is open and $f^{\varphi}(b_1, \ldots, b_k)$ is a non-interior point of $[\varphi(x, b_1, \ldots, b_k)]^{(\mathfrak{B}_\beta, r_\beta)}$.

Take $(\mathfrak{B}_\alpha, r_\alpha)$ to be a model of T_α of cardinality κ where r_α is the set of definable open sets. The purpose of $(\mathfrak{B}_\alpha, r_\alpha)$ is to enable us to take infinite intersections of open sets and to make them open.

If α is a limit ordinal then we take $(\mathfrak{B}_\alpha, r_\alpha)$ to be the union of the elementary chain $(\mathfrak{B}_\beta, r_\beta)$, $\beta < \alpha$.

Let $(\mathfrak{B}, r)$ be the union of the elementary chain $(\mathfrak{B}_\alpha, r_\alpha)$, $\alpha < \kappa$. Again by Theorem 1.4 $(\mathfrak{B}_\alpha, r_\alpha) \prec (\mathfrak{B}, r)$ for $\alpha < \kappa$.

Define r^* to be the topology generated by $\{\mathcal{O}_{(b,\beta)} \mid b \in B, \beta < \kappa\}$ where $\mathcal{O}_{(b,\beta)} = \bigcap_{\mathcal{O} \in \mathscr{E}(b,\beta)} \mathcal{O}$ and $\mathscr{E}(b, \beta)$ is $\{\mathcal{O} \subseteq B \mid b \in \mathcal{O}$ and $\mathcal{O}$ is a definable clopen set of r with parameters from $(\mathfrak{B}_\beta, r_\beta)\}$.

We claim that $(\mathfrak{B}, r) \equiv (\mathfrak{B}, r^*)$. This is most easily shown by induction on the complexity of the formulas with parameters in B. The difficult case is the Q clause. Since $r \subseteq r^*$ we have that if $(\mathscr{L}, r) \models Qx\chi(x)$ then $(\mathfrak{B}, r^*) \models Qx\chi(x)$. Suppose that $(\mathfrak{B}, r) \models \sim Qx\chi(x)$. Then if $[\chi(x)]^{(\mathfrak{B}, r)} = [\chi(x)]^{(\mathfrak{B}, r^*)} = \bigcup_{t \in T} \mathcal{O}_{(b_t, \beta_t)}$, we have $f^{\varphi}(b_1, \ldots, b_k) \in \mathcal{O}_{(b_j, \beta_j)}$ for some $j \in T$. However, $\mathcal{O}_{(b_j, \beta_j)} = \bigcap_{\gamma \in G} [\psi_\gamma(x)]^{(\mathfrak{B}, r)}$. Thus for $\theta < \kappa$, θ a sufficiently large limit ordinal, we have from the definition of T_α, $\alpha < \kappa$, that $(\mathfrak{B}_\theta, r_\theta) \models \sim \chi(c_\varphi) \wedge \bigwedge_{\gamma \in G} \psi_\gamma(c_\varphi)$. Hence $(\mathfrak{B}, r^*) \models \sim Qx\chi(x)$.

$(\mathfrak{B}, r^*)$ is 0-dimensional and regular since $(\mathfrak{B}, r^*)$ has a clopen basis of cardinality κ which is closed under intersections of cardinaltiy less than κ. This is because κ is regular and the definition of the T_α's and r^*. We show that $(\mathfrak{B}, r^*)$ is normal by using a generalization of Theorem 18.14, ([2], p. 121), as follows:

Theorem 3.1.7. *Let (X, τ) be a regular topological space of cardinality κ, κ regular. Then if τ has a basis of cardinality κ which is closed under intersections of cardinality less than κ then it is normal. In fact it is paracompact,* ([2], p. 338).

This theorem has the following interesting corollary.

Corollary 3.1.8. *Let T be a countable* L(Q) *theory. Then T is consistent in* L(Q) *with $\forall x\, Qy(x \neq y)$ if and only if T has a second countable 0-dimensional metrizable topological model.*

Proof. Easy by the fact that a second countable, regular, and fréchet space is metrizable.

3.2

In this section we shall prove that the quantifier Q_{ω_1} which has the interpretation "there exists uncountably many" is compatible with the topological quantifier Q. $L(Q_{\omega_1})$ has been extensively studied in [6]. We shall only present here the basic completeness theorem without proof. The axioms for $L(Q_{\omega_1})$ are:

(B0) the axiom schemes for L(Q) (excluding (A1)–(A4)),

(B1) $\forall x\, (\varphi \to \psi) \to (Q_{\omega_1} x\varphi \to Q_{\omega_1} x\psi)$,

(B2) $Q_{\omega_1} x(x = y \vee x = z)$,

(B3) $Q_{\omega_1} y\, \exists x\, \varphi(x, y) \to \exists x\, Q_{\omega_1} y\varphi(x, y) \vee Q_{\omega_1} x\, \exists y\, \varphi(x, y)$.

Throughout this section we will assume that L is countable.

$L(Q_{\omega_1}, Q)$ is defined in the natural way as the composition of $L(Q_{\omega_1})$ and L(Q) (i.e. $L(Q_{\omega_1}, Q)$ has (A0)–(A4) and (B0)–(B3) as its axioms).

Definition 3.2.1. Let $(\mathfrak{A}, q_{\omega_1}, q)$ be a weak model for $L(Q_{\omega_1}, Q)$. Then $(\mathfrak{A}, q_{\omega_1}, q)$ is called *standard* if q_{ω_1} is the set of uncountable subsets of A.

With this definition we are now in a position to state a weak completeness theorem for $L(Q_{\omega_1}, Q)$ whose proof is essentially the same as the proof for $L(Q_{\omega_1})$ found in Keisler [6].

Theorem 3.2.2 (Keisler). *Let T be an* $L(Q_{\omega_1}, Q)$ *theory. Then T is consistent with* (B0)–(B3) *if and only if T has a standard (not necessarily topological) model.*

Now we are able to state and prove the main theorem of this section.

Theorem 3.2.3. *Let T be an* $L(Q_{\omega_1}, Q)$ *theory. T is consistent in* $L(Q_{\omega_1}, Q)$ *if and only if T has a standard topological model.*

Proof. (if direction). This follows since the axioms of $L(Q_{\omega_1}, Q)$ are true in every standard topological model.

(only if direction). Assume that T is consistent in $L(Q_{\omega_1}, Q)$. Define $L' = L \cup \{f^\varphi \mid \varphi \in L(Q_{\omega_1}, Q)\}$ and $T' = T \cup \{\psi^\varphi \mid \psi, \varphi \in L(Q_{\omega_1}, Q)\}$. Recall that f^φ and ψ^φ were defined in Section 2 in the proof of the basic completeness theorem for $L(Q)$.

We will now show that T' is consistent with the axiom schemas (B0)–(B3). Since T is $L(Q_{\omega_1}, Q)$ consistent, using the completeness theorem for $L(Q_{\omega_1}, Q)$ we obtain a standard, not necessarily topological, model, $(\mathfrak{A}, q_{\omega_1}, q)$, where q is the set of definable open sets (with parameters). Define f^{φ_i} $0 \leq i \leq m$ as in Section 2 and also take $\psi_i^{\varphi_i}$ $0 \leq i \leq m$ as in Section 2. Then one obtains $(\mathfrak{A}, q_{\omega_1}, q, f^{\varphi_1}, \ldots, f^{\varphi_n}) \models T \cup \{\psi_i^{\varphi_i}\}_{1 \leq i \leq m}$. Using the compactness theorem for $L'(Q_{\omega_1}, Q)$ we see that T' is consistent with (B0)–(B3).

Applying the completeness theorem for $L'(Q_{\omega_1}, Q)$ we have a standard model, $(\mathfrak{A}, q_{\omega_1}, q)$ of T' where q is the set of definable open sets of $L'(Q_{\omega_1}, Q)$ which again is not necessarily topological.

Let q^* be the topology generated by the $L(Q_{\omega_1}, Q)$ definable open sets. As in the basic completeness theorem using the f^φ's and ψ^φ's we obtain:

$$(\mathfrak{A}, q_{\omega_1}, q) \underset{L(Q_{\omega_1}, Q)}{\equiv} (\mathfrak{A}, q_{\omega_1}, q^*).$$

So we have produced a standard topological model.

Definition 3.2.4. Let $L(Q_{\omega_\alpha})$ be the logic with the new quantifier symbol Q_{ω_α} whose interpretation is "there exists at least ω_α many". $L(Q_{\omega_\alpha}, Q)$ is defined analogously. Also a standard $L(Q_{\omega_\alpha}, Q)$ model is defined analogously to a standard topological model.

Thus we have the following result

Corollary 3.2.5. (i) *Assume ω_α is regular. If a sentence θ is valid under the* $L(Q_{\omega_1}, Q)$ *interpretation then it is valid under the* $L(Q_{\omega_\alpha}, Q)$ *interpretation.*

(ii) *Assume the* GCH. *Then θ has an* $L(Q_{\omega_1}, Q)$ *model if and only if it has a* $L(Q_{\omega_\alpha}, Q)$ *model.*

Proof. $\{\theta\} \cup \{\psi^\varphi \mid \psi, \varphi \in L(Q_{\omega_1}, Q)\}$ is $L(Q_{\omega_\alpha})$ consistent This follows from the analogous problem for $L(Q_{\omega_\alpha})$ found in [6] which uses the two cardinal results of Chang and Vaught.

One might ask the question whether or not $L(Q_{\omega_1}, Q)$ is more powerful with respect to topology than say $L(Q)$. The following results are one step in that direction. First, however, we need to review several topological definitions.

Definition 3.2.6. A topological space (X, τ) is called *second countable* if there is a countable set, S, of open sets which generate τ.

Definition 3.2.7. A topological space (X, τ) is called *first countable* if for each point x, there is a countable set S_x of open sets each of whose members contains x such that for every open set $\mathcal{O}$ containing x, $\mathcal{O}$ also contains a member of S_x.

Definition 3.2.8. A topological space (X, τ) is called *separable* if it contains a countable set S of points whose closure is X, (i.e. $\bar{S} = X$).

Since we have a means of distinguishing countable from uncountable it seems natural to ask whether there is a completeness theorem for separable, first countable, or second countable topological models in $L(Q_{\omega_1}, Q)$. The answer is yes. However, we have not been able to give an explicit axiomatization but one which requires the introduction of a new predicate symbol. This gives us a method for deciding whether or not a sentence is consistent with the theory but not a simple natural set of axioms.

Theorem 3.2.9. *Let T be an $L(Q_{\omega_1}, Q)$ theory.*

(a) *Let $U(x, y)$ be a new binary predicate symbol and $L' = L \cup \{U(x, y)\}$. T has a standard second countable topological model if and only if T is consistent in $L'(Q_{\omega_1}, Q)$ with:*

$$\forall y\, QxU(x, y) \wedge \sim Q_{\omega_1} y\, \exists x\, U(x, y). \tag{1}$$

That is $U(x, y)$ defines a countable collection of open sets.

$$\begin{aligned} &\forall z\, \forall z_1, \dots, z_n (Qx\varphi(x, z_1, \dots, z_n) \wedge \varphi(z, z_1, \dots, z_n) \\ &\quad \rightarrow \exists y\, \forall x\, ((U(x, y) \rightarrow \varphi(x, z_1, \dots, z_n)) \wedge U(z, y))). \end{aligned} \tag{2}$$

This is equivalent to saying that the collection of open sets defined by U forms a basis for the topology generated by the definable open sets.

(b) *Let $U(x, y, z)$ be a new ternary predicate symbol and $L' = L \cup \{U(x, y, z)\}$. T has a standard first countable topological model if and only if T is consistent in $L'(Q_{\omega_1}, Q)$ with:*

$$\forall y\, \forall z\, QxU(x, y, z) \wedge \forall z \sim Q_{\omega_1} y\, \exists x\, U(x, y, z) \tag{3}$$

$U(x, y, z)$ forms a countable collection of open sets around each point.

$$\begin{aligned} &\forall z\, \forall z_1, \dots, z_n (Qx\varphi(x, z_1, \dots, z_n) \wedge \varphi(z, z_1, \dots, z_n) \\ &\quad \rightarrow \exists y\, \forall x\, ((U(x, y, z) \rightarrow \varphi(x, z_1, \dots, z_n)) \wedge U(z, y, z))). \end{aligned} \tag{4}$$

$U(x, y, z)$ defines a local basis around each point for the topology generated by the definable open sets.

(c) *Let $U(x)$ be a new unary predicate symbol and $L' = L \cup \{U(x)\}$*

T has a standard separable topological model if and only if T is consistent in $\mathrm{L}'(\mathrm{Q}_{\omega_1}, \mathrm{Q})$ with:

$$\sim \mathrm{Q}_\omega x U(x). \tag{5}$$

U defines a countable set of points.

$$\forall z_1, \ldots, z_n (\mathrm{Q}x\varphi(x, z_1, \ldots, z_n) \to \exists x\, (\varphi(x, z_1, \ldots, z_n) \wedge U(x))). \tag{6}$$

That is U defines a set of points which are dense in the topology generated by the definable open sets.

Proof. These are all easy consequences of the $\mathrm{L}(\mathrm{Q}_{\omega_1}, \mathrm{Q})$ completeness theorem and the topological definitions.

Now that we have demonstrated that the theories of second countable, separable, and first countable topological models are axiomatizable an obvious question is whether or not their $\mathrm{L}(\mathrm{Q}_{\omega_1}, \mathrm{Q})$ axiomatizations differ.

It is easy to see that the axiomatization for first countable spaces is not the same as for separable or second countable since $\mathrm{Q}_{\omega_1}x(x = x) \wedge \forall y\, \mathrm{Q}x(x = y)$ is true in an uncountable discrete topological model. However it has no separable or second countable topological model.

Since every second countable topological model is separable all we need to show is that there is a sentence φ which has a separable topological model but no second countable model.

Let L be $\{=, <\}$. Now we will construct φ. Let φ be:

$$(< \text{ is a linear order without end points}) \wedge$$
$$\wedge\, \mathrm{Q}_{\omega_1}x(x = x) \wedge \forall y\, \forall x\, \mathrm{Q}z(x \leq z < y)$$

i.e. $<$ is a linear ordering on an uncountable space and the "half open" intervals are open. The real line with the half open interval topology is uncountable and separable.

We will show that it does not have an uncountable second countable topological model. Suppose $(\mathfrak{A}, \mathrm{q}_{\omega_1}, \mathrm{q})$ is a standard topological model of φ. Then suppose q has a countable basis, $\{\mathcal{O}_i\}_{i \in \omega}$.

Let $\mathrm{Inf}(\mathcal{O}_i)$ be the least z in $\mathcal{O}_i$ if it exists. Take $t \notin \{\mathrm{Inf}(\mathcal{O}_i) \mid \mathrm{Inf}(\mathcal{O}_i) \text{ exists}\}$. This is possible since A is uncountable.

We claim that $[t, z)$ does not contain any $\mathcal{O}_i$ which contains t. If it did then $\mathrm{Inf}(\mathcal{O}_i) = t$ which is a contradiction. Thus $(\mathfrak{A}, \mathrm{q}_{\omega_1}, \mathrm{q})$ is not second countable.

3.3

Using the weak completeness theorem for $\mathrm{L}_{\omega_1\omega}(\mathrm{Q})$ in Section 1 we can give a completeness theorem for $\mathrm{L}_{\omega_1\omega}(\mathrm{Q})$ which is the infinitary logic formed by combining $\mathrm{L}_{\omega_1\omega}$ with the quantifier symbol Q. $\mathrm{L}_{\omega_1\omega}$ is the infinitary logic formed by allowing countably infinite conjunctions but only finite quantifiers.

In $L_{\omega_1\omega}(Q)$ the notion of $(\mathfrak{A}, q) \models \varphi[a_1, \dots, a_n]$ is defined in the obvious way.

The axioms for $L_{\omega_1\omega}(Q)$ are very straightforward and are an adaptation to $L_{\omega_1\omega}(Q)$ of those found in [6].

(I) Axioms of $L(Q)$,

(II) $\bigwedge_{n\in\omega}(\varphi \to \psi_n) \to (\varphi \to \bigwedge_{n\in\omega}\psi_n)$,

(III) $(\bigwedge_{n\in\omega}\psi_n) \to \psi_m$, $m \in \omega$,

(IV) $\bigwedge_{n\in\omega} Qx\psi_n(x) \to Qx \bigvee_{n\in\omega}\psi_n(x)$.

The rules of inference are modus ponens, generalization and the following infinite rule:

From $\psi_0, \psi_1, \psi_2, \dots$, infer $\bigwedge_{n\in\omega}\psi_n$.

We thus are able to prove the following.

Theorem 3.3.1. *A sentence φ of $L_{\omega_1\omega}(Q)$ is consistent in $L_{\omega_1\omega}(Q)$ if and only if φ has a topological model.*

Proof. (if direction). Easy since all the axioms of $L_{\omega_1\omega}(Q)$ are true in every topological model.

(only if direction). Assume φ is consistent in $L_{\omega_1\omega}(Q)$. Since φ is a sentence each subformula of φ has only finitely many free variables and moreover φ has only countable many subformulas.

For each subformula $\psi(x_1, \dots, x_n)$ of φ introduce a new predicate symbol R_ψ with n-places forming an expanded language L'. Let Γ be the set of sentences of $L'_{\omega_1\omega}(Q)$ consisting of φ plus the sentences

$$(\forall x_1, \dots, x_n)(\psi(x_1, \dots, x_n) \leftrightarrow R_\psi(x \dots, x_n))$$

for each subformula ψ $(x_1, \dots, x_n)$ of φ.

It is not hard to show that Γ is consistent in $L'_{\omega_1\omega}(Q)$ for any deduction of a contradiction from Γ can be made into a contradiction from φ in $L_{\omega_1\omega}(Q)$ by eliminating the symbols R_ψ in favor of ψ.

For each ψ $(x, y_1, \dots, y_n) \in L'(Q)$ let ψ^* be the sentence

$$\forall y_1, \dots, y_n \, \exists x \, (\sim Qx\psi(x, y_1, \dots, y_n) \to \bigwedge_{\varphi(x, z_1, \dots, z_{n_\varphi}) \in L'(Q)} \psi^\varphi(x, y_1, \dots, y_n))$$

where $\psi^\varphi(x, y_1, \dots, y_n)$ is:

$$\forall z_1, \dots, z_{n_\varphi} (Qx\varphi(x, z_1, \dots, z_n) \wedge \forall x \, (\varphi(x, z_1, \dots, z_{n_\varphi})$$
$$\to \psi(x, y_1, \dots, y_n)) \to (\psi(x, y_1, \dots, y_n) \wedge \sim \varphi\,(x, z_1, \dots, z_{n_\varphi}))).$$

Now $\psi^\varphi(x, y_1, \dots, y_n)$ is just the set of x which are in $\psi(x, y_1, \dots, y_n)$ but not in any open subset of ψ defined by φ using parameters. Thus ψ^* means that for any parameters, if ψ is not open then it is not equal to any union of open sets definable in $L'(Q)$.

We claim that $\Gamma \cup \{\psi^* \mid \psi \in L'(Q)\}$ is $L'_{\omega_1\omega}(Q)$ consistent. We will show that axiom (IV) for $L_{\omega_1\omega}(Q)$ implies ψ^*. First we have

$\forall y_1, \ldots, y_n(\bigwedge_{\varphi(x, z_1, \ldots, z_{n_\varphi}) \in L'(Q)} Qx\psi^\varphi(x, y_1, \ldots, y_n))$ which by axiom (IV) gives $\forall y_1, \ldots, y_n Qx(\bigvee_{\varphi(x, z_1, \ldots, z_{n_\varphi}) \in L'(Q)} \psi^\varphi(x, y_1, \ldots, y_n))$. This then yields ψ^*.

Using the weak completeness theorem for $L'_{\omega_1\omega}(Q)$ given in Section 1 we obtain a weak model

$$(\mathfrak{A}, q) \models \Gamma \cup \{\psi^* \mid \psi \in L'(Q)\}.$$

Let q^* be the topology generated by the open sets $L'(Q)$ definable in $(\mathfrak{A}, q)$ with parameters in A. By an argument analogous to that used to prove the basic completeness theorem for $L(Q)$ we obtain

$$(\mathfrak{A}, q) \underset{L'(Q)}{\equiv} (\mathfrak{A}, q^*).$$

Carefully examining the definition of $L'(Q)$ we see that Γ implies that φ is equivalent to a sentence of $L'(Q)$. Therefore, $(\mathfrak{A}, q^*) \models \varphi$ and thus we have a topological model of φ.

4. Counterexamples and final comments

We conclude this paper by presenting counterexamples to the $L(Q)$ analogues of interpolation and definability. We also present some preliminary work into the theory of ultraproducts of topological models. This work consists of a proof of a Łoś type theorem for ultraproducts of topological models.

In 4.4 we make some concluding remarks regarding our current research into the axiomatization of product topologies and continuous functions. As an application of these results we can obtain an axiomatization of the $L(Q)$ theory of topological groups.

4.1

In this section we will show that in contrast to the first order predicate calculus, $L(Q)$ does not satisfy the interpolation problem. The formulation of the interpolation problem which we will use is as follows: Let ψ, φ be sentences of $L(Q)$ such that $\models \varphi \to \psi$. Then there is a sentence θ of $L(Q)$ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$ and every relation, function or constant symbol of $L(Q)$ which occurs in θ occurs in both φ and ψ.

We will construct sentences φ and ψ of languages L_1, L_2 respectively and topological models $(\mathfrak{A}, q_1)$ and $(\mathfrak{A}, q_2)$ such that $\models \varphi \to \psi$, $(\mathfrak{A} \restriction L_1 \cap L_2, q_1) \equiv (\mathfrak{A} \restriction L_1 \cap L_2, q_2)$, $(\mathfrak{A}, q_1) \models \varphi$ and $(\mathfrak{A}, q_2) \models \sim\psi$. This is clearly seen to violate the interpolation problem since any interpolant is in $L_1 \cap L_2(Q)$ and the above models are elementary equivalent in $L_1 \cap L_2(Q)$. Thus if $(\mathfrak{A}, q_2) \models \sim\psi$ we obtain that $(\mathfrak{A}, q_2) \models \sim\theta$ but then we also have $(\mathfrak{A}, q_1) \models \theta$, which is a contradiction.

Let $L_1 = \{B(x), C(x), R(x)\}$ and $L_2 = \{B(x), C(x), P(x)\}$ then we let $\varphi(R)$ be

$$\sim QxB(x) \wedge \forall y\,(B(y) \leftrightarrow C(y) \vee R(y)) \wedge QxR(x)$$

which says that B is not open, R is open and B is the union of R and C. Take $\psi(P)$ to be

$$\forall x\,(C(x) \rightarrow P(x)) \rightarrow \sim Qx(P(x) \wedge B(x))$$

which says that P contains C and $P \cap B$ is not open. Then one easily sees that $\models \varphi(r) \rightarrow \psi(P)$ since otherwise E would be equal to R union P. Since a union of open sets is open, B would be open.

Let

$$A = \mathbf{N} \text{ i.e. set of natural numbers,}$$

$$B^{\mathfrak{A}} = \{2n \mid n \in \mathbf{N}\},$$

$$C^{\mathfrak{A}} = \{4n \mid n \in \mathbf{N}\}.$$

There are many choices for $R^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$. For example let $R^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$ be:

$$R^{\mathfrak{A}} = \{n \mid n \in B^{\mathfrak{A}} - C^{\mathfrak{A}} \text{ or } n = 8k \text{ for some } k \in \mathbf{N}\}$$

$$P^{\mathfrak{A}} = \{n \mid n \in C^{\mathfrak{A}} \text{ or } n \in B^{\mathfrak{A}} \text{ and } n = 4k+1 \text{ for some } k \in \mathbf{N}\}.$$

Now define $(\mathfrak{A}, \mathrm{q}_1)$ to be $\langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, R^{\mathfrak{A}}, P^{\mathfrak{A}}, \{\mathbf{N}, \emptyset, R^{\mathfrak{A}}\}\rangle$ and let $(\mathfrak{B}, \mathrm{q}_1) = \langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, \{\mathbf{N}, \emptyset, R^{\mathfrak{A}}\}\rangle$. Take $(\mathfrak{A}, \mathrm{q}_2)$ to be $\langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, R^{\mathfrak{A}}, P^{\mathfrak{A}}, \{\mathbf{N}, \emptyset, P^{\mathfrak{A}}\}\rangle$ and let $(\mathfrak{B}, \mathrm{q}_2) = \langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, \{\mathbf{N}, \emptyset, P^{\mathfrak{A}}\}\rangle$.

It is easily seen that

$$(\mathfrak{A}, \mathrm{q}_1) \models \varphi(R)$$

and

$$(\mathfrak{A}, \mathrm{q}_2) \models \sim \psi(P).$$

Now if we can show that $(\mathfrak{B}, \mathrm{q}_1) \equiv (\mathfrak{B}, \mathrm{q}_2)$ then we are done.

Lemma 4.1. *Let* $(\mathfrak{B}, \mathrm{q}_1)$ *and* $(\mathfrak{B}, \mathrm{q}_2)$ *be as above then* $(\mathfrak{B}, \mathrm{q}_1) \equiv (\mathfrak{B}, \mathrm{q}_2)$.

Proof. Suppose that we can prove that:

$$(*) \qquad (\mathfrak{B}, \mathrm{q}_1), (\mathfrak{B}, \mathrm{q}_2) \models \forall y_1, \ldots, y_n \{(Qx\varphi(x, y_1, \ldots, y_n)) \leftrightarrow (\forall x\, \varphi(x, y_1, \ldots, y_n) \vee \forall x \sim \varphi(x, y_1, \ldots, y_n))\}$$

where $\varphi(x, y_1, \ldots, y_n)$ is a formula of L. Then by induction on the occurrences of Q we can easily show that for each $\psi(y_1, \ldots, y_n)$, a formula of L(Q), there is a $\varphi(y_1, \ldots, y_n)$ of L such that

$$(\mathfrak{B}, \mathrm{q}_1), (\mathfrak{B}, \mathrm{q}_2) \models \forall y_1, \ldots, y_n\, (\varphi(y_1, \ldots, y_n) \leftrightarrow \psi(y_1, \ldots, y_n))$$

which easily implies that

$(\mathfrak{B}, q_1) \equiv (\mathfrak{B}, q_2)$.

To prove (*) we need only to show that $R^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$ are not definable with parameters over $(\mathfrak{B}, q_1)$ or $(\mathfrak{B}, q_2)$ respectively. We will prove that $R^{\mathfrak{A}}$ is not definable with parameters over $(\mathfrak{B}, q_1)$ and the proof for $P^{\mathfrak{A}}$ over $(\mathfrak{B}, q_2)$ is analogous.

Suppose $R^{\mathfrak{A}}$ is defined in $(\mathfrak{B}, q_1)$ by $\psi(x, a_1, \ldots, a_n)$. Then we notice that there is a bijective map, f, from A onto A which keeps $a_1, \ldots, a_n$, $B^{\mathfrak{A}}$, $C^{\mathfrak{A}}$ fixed but moves some element of $R^{\mathfrak{A}}$ to one element outside of $R^{\mathfrak{A}}$. Thus $R^{\mathfrak{A}}$ is not definable by a formula of L with parameters in $(\mathfrak{B}, q_1)$. So we have proved the lemma and thus obtain the counterexample.

4.2

In this section we present a counterexample to the definability problem for L(Q). This counterexample uses the counterexamples constructed in 4.1 and a modification of a construction used by Makowsky and Shelah in [7]. Their method converts a counterexample to interpolation into a counterexample to the definability problem.

We will use the following form of the definability problem in our counterexample. Let L be a language, and P and R be new predicate symbols. If $\Sigma(X)$ is a set of formulas with non-logical parameters from L such that $\Sigma(P) \cup \Sigma(R) \vdash P \leftrightarrow R$ then there is a $\theta \in L(Q)$ so that $\Sigma(P) \vdash P \leftrightarrow \theta$. That is if a predicate is implicitly definable then it is explicitly definable.

Let $\varphi(P)$ and $\psi(P)$ be the sentences constructed in the interpolation counterexample. Also let $f(x)$ be a new function symbol. We then define $\Sigma(P)$ to be

$$\{\forall x\, (\varphi^{f^{-1}(x)}(P) \vee \psi^{f^{-1}(x)}(P)),\ \forall x\, (P(x) \leftrightarrow \varphi^{f^{-1}(x)}(P))\}$$

where $\varphi^{f^{-1}(x)}(P)$ is the relativization of $\varphi(P)$ to $f^{-1}(x)$. Notice that by the construction of $\varphi(P)$ and $\psi(P)$ we have $\sim (\varphi^{f^{-1}(x)}(P) \wedge \psi^{f^{-1}(x)}(R))$ for any P and R. This follows from the fact that the union of open sets is open.

We claim that $\Sigma(P) \cup \Sigma(R) \vdash P \leftrightarrow R$. Suppose not. Then there is a topological model $(\mathfrak{A}, P^{\mathfrak{A}}, R^{\mathfrak{A}}, q)$ of $\Sigma(P) \cup \Sigma(R)$ which also models $\exists x\, (\sim P(x) \wedge R(x))$. Hence there is an $a \in (R^{\mathfrak{A}} - P^{\mathfrak{A}})$ such that

$$(\mathfrak{A}, P^{\mathfrak{A}}, R^{\mathfrak{A}}, q) \models (\varphi^{f^{-1}(a)}(R) \vee \psi^{f^{-1}(a)}(R)),$$

$$(\varphi^{f^{-1}(a)}(P) \vee \psi^{f^{-1}(a)}(P)),$$

$$(R(a) \rightarrow \varphi^{f^{-1}(a)}(R))$$

and

$$(\sim P(a) \rightarrow \sim \varphi^{f^{-1}(a)}(P)).$$

Thus $(\varphi^{f^{-1}(a)}(R) \wedge \psi^{f^{-1}(a)}(P))$ which is a contradiction by an earlier remark. We have consequently shown that $\Sigma(P)$ implicitly defines P.

Now we need to demonstrate that P is not explicitly definable to obtain the counterexample. We will construct a model of $\Sigma(P)$, using the models of $\varphi(P)$ and $\psi(P)$ constructed in 4.1, where P is not explicitly definable in it.

Take

$$|\mathcal{N}| = \{\langle x_1, \dots, x_n\rangle \mid x_i \in N \text{ or it is } \vec{0}\}$$

$$C^{\mathcal{N}} = \{\langle x_1, \dots, x_n\rangle \mid x_n \in C^{\mathfrak{A}} \text{ or it is } \vec{0}\}$$

$$B^{\mathcal{N}} = \{\langle x_1, \dots, x_n\rangle \mid x_n \in B^{\mathfrak{A}} \text{ or it is } \vec{0}\}$$

$$P^{\mathcal{N}} = \{\langle x_1, \dots, x_n\rangle \mid x_i \in P^{\mathfrak{A}} \text{ for all } i \text{ or it is } \vec{0}\}$$

$$R^{\mathcal{N}} = \{\langle x_1, \dots, x_n\rangle \mid x_n \in R^{\mathfrak{A}} \text{ and } x_{n-1} \notin P^{\mathfrak{A}} \text{ or it is } \vec{0}\}$$

$$f^{\mathcal{N}} = (\langle x_1, \dots, x_n\rangle) = \langle x_1, \dots, x_{n-1}\rangle$$

$$f^{\mathcal{N}}(\langle x\rangle) = \vec{0}$$

$$f^{\mathcal{N}}(\emptyset) = \vec{0}.$$

Finally $\mathcal{N} = \langle |\mathcal{N}|, C^{\mathcal{N}}, B^{\mathcal{N}}, P^{\mathcal{N}}, R^{\mathcal{N}}, f^{\mathcal{N}}\rangle$ and define q to be the topology generated by:

$$\{|\mathcal{N}|, \emptyset, P^{\mathcal{N}} \cap f^{-1}(x), R^{n} \cap f^{-1}(x) \text{ for each } x \in |\mathcal{N}|\}.$$

Now $(\mathcal{N}, \mathrm{q}) \models \Sigma(P)$ since $f^{-1}(x)$ is isomorphic to either $(\mathfrak{A}, \mathrm{q}_1)$ or $(\mathfrak{A}, \mathrm{q}_2)$ which are the models constructed in the counterexample to the interpolation problem. Hence $P^{\mathcal{N}}$ is implicitly definable in $(\mathcal{N}, \mathrm{q})$.

We claim that $P^{\mathcal{N}}$ is not explicitly definable with parameters in $(\mathcal{N}, \mathrm{q})$. The easiest method to show this is to prove for each $\theta(x, y, \dots, y) \in \mathrm{L(Q)}$ we have

$$(\mathcal{N}, \mathrm{q}) \models \forall y_1, \dots, \forall y_n\, (\mathrm{Q}x\theta(x, y_1, \dots, y_n)$$
$$\leftrightarrow (\forall x\, \theta(x, y_1, \dots, y_n) \vee \forall x \sim \theta(x, y_1, \dots, y_n))).$$

This is to say that only the trivial open sets are definable. As usual we will prove this by induction on the number of occurrences of the quantifier symbol Q. It is straightforward to show that if we prove the claim for all θ in L then it holds for all ψ in L(Q) since it provides a method for reducing the number of occurrences of Q.

Suppose that another set in q is definable by a formula of L with parameters. Then $P^{\mathcal{N}} \cap f^{-1}(x)$ is definable by a formula of L with parameters. Take a permutation, g, of $|\mathcal{N}|$ which leaves fixed $C^{\mathcal{N}}, B^{\mathcal{N}}, a_1, \dots, a_n$ but moves $P^{\mathcal{N}} \cap f^{-1}(x)$. Since g is an automorphism of $\mathcal{N}$ which moves $P^{\mathcal{N}} \cap f^{-1}(x)$, it is not definable by a formula of L with parameters. By the remark above only the trivial open sets are definable. Consequently, $P^{\mathcal{N}}$ is not explicitly definable and we have constructed a counterexample to the definability problem.

4.3

In this section we will define the notion of an ultraproduct of topological models. The only difficulty is in the definition for the Q clause.

Definition 4.3.1. Let $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha \in I$, be a collection of topological models and $\mathcal{U}$ an ultrafilter on I. We define the ultraproduct of $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha \in I$, in symbols $\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, as follows:

$$\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) = (\Pi_{\mathcal{U}} \mathfrak{A}_\alpha, \Pi_{\mathcal{U}} \mathrm{q}_\alpha)$$

where $\Pi_{\mathcal{U}} \mathfrak{A}_\alpha$ is just the regular ultraproduct for first order logic and $\Pi_{\mathcal{U}} \mathrm{q}_\alpha$ = the topology generated by

$$\{[\Pi_{\mathcal{U}} \mathcal{O}_\alpha]_{\mathcal{U}} \mid \{\alpha \mid \mathcal{O}_\alpha \in \mathrm{q}_\alpha\} \in \mathcal{U}\},$$

where

$$[\Pi_{\mathcal{U}} \mathcal{O}_\alpha]_{\mathcal{U}} = \{f \in \Pi_{\mathcal{U}} A_\alpha \mid \{\beta \mid f(\beta) \in \mathcal{O}_\beta\} \in \mathcal{U}\}.$$

This definition enables us to prove the following interesting theorem.

Theorem 4.3.2 (Łoś Theorem for Ultraproducts). *Let $(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$, $\alpha \in I$, be a collection of topological models and $\mathcal{U}$ an ultrafilter on I. Then for any formula $\varphi(f_1, \ldots, f_n)$ of* L(Q) *with parameters in $\Pi_{\mathcal{U}} A_\alpha$ we obtain:*

$$\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \varphi[f_1, \ldots, f_n] \text{ if and only if}$$
$$\{\alpha \mid (\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \varphi[f_1(\alpha), \ldots, f_n(\alpha)]\} \in \mathcal{U}.$$

Proof. The proof is by induction on the complexity of φ. As usual the only difficult case is the Q clause.

By the induction hypothesis we obtain

$$[\varphi(x, f_1, \ldots, f_n)]^{\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)} = [\Pi_{\mathcal{U}}[(\varphi(x, f_1(\beta), \ldots, f_n(\beta)))]^{(\mathfrak{A}_\beta, \mathrm{q}_\beta)}].$$

Assume $\{\beta \mid (\mathfrak{A}_\beta, \mathrm{q}_\beta) \models \mathrm{Q}x\varphi(x, f_1(\beta), \ldots, f_n(\beta))\} \in \mathcal{U}$. We will show that $\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \mathrm{Q}x\varphi(x, f_1, \ldots, f_n)$. But $\{\beta \mid [\varphi(x, f_1(\beta), \ldots, f_n(\beta))]^{(\mathfrak{A}_\beta, \mathrm{q}_\beta)} \in \mathrm{q}_\beta\} \in \mathcal{U}$. Thus by the definition of $\Pi_{\mathcal{U}} \mathrm{q}_\alpha$ we obtain $[\varphi(x, f_1, \ldots, f_n)]^{\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)} \in \Pi_{\mathcal{U}} \mathrm{q}_\alpha$ so $\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models Qx\varphi(x, f_1, \ldots, f_n)$.

Now assume that $\{\alpha \mid (\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \mathrm{Q}x\varphi(x, f_1(\alpha), \ldots, f_n(\alpha))\} \not\in \mathcal{U}$. Then we have that $\{\alpha \mid (\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \sim Qx\varphi(x, f_1(\alpha), \ldots, f_n(\alpha))\} \in \mathcal{U}$. Thus we need only to show that $\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \sim Qx\varphi(x, f_1, \ldots, f_n)$.

We will do this by defining a $c \in \Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)$ which will insure that fact. For some $V \in \mathcal{U}$ we have that $\{\alpha \mid (\mathfrak{A}_\alpha, \mathrm{q}_\alpha) \models \sim \mathrm{Q}x\varphi(x, f_1(\alpha), \ldots, f_n(\alpha))\} = V$. Thus, for each $\alpha \in V$ we let $c(\alpha)$ be an element of $[\varphi(x, f_1(\alpha), \ldots, f_n(\alpha))]^{(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)}$ which is not in any open subset (i.e. $c(\alpha)$ is not an interior point). For $\alpha \not\in V$ let $c(\alpha)$ be arbitrary.

Now if $[\varphi(x, f_1, \ldots, f_n)]^{\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)} \in \Pi_{\mathcal{U}} \mathrm{q}_\alpha$ we obtain that $[\varphi(x, f_1, \ldots, f_n)]^{\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)} = \bigcup_{t \in T}[\Pi_{\mathcal{U}} \mathcal{O}_{\alpha,t}]$. Also $c \in [\varphi(x, f_1, \ldots, f_n)]^{\Pi_{\mathcal{U}}(\mathfrak{A}_\alpha, \mathrm{q}_\alpha)}$. But for each $t \in T$, $c \not\in [\Pi_{\mathcal{U}} \mathcal{O}_{\alpha,t}]$, since if it were, we would have $\{\alpha \mid c(\alpha) \in \mathcal{O}_{\alpha,t}\} \in \mathcal{U}$, which implies that there is an $\alpha \in V$, such that $c(\alpha)$ is an interior point. This gives a contradiction. So, $\Pi_{\mathcal{U}}(\mathfrak{A}, \mathrm{q}_\alpha) \models \sim \mathrm{Q}x\varphi(x, f_1, \ldots, f_n)$.

4.4

In this section the author wishes to make several remarks concerning the axiomatization of product topologies and continuous functions. Let $L(Q^n_{n\in\omega})$ be the language formed by adding the new quantifier symbols Q^n for each $n \in \omega$. The intended interpretation of $Q^n x_1, \ldots, x_n \varphi(x_1, \ldots, x_n)$ is that the set defined by $\varphi(x_1, \ldots, x_n)$ is "open in the nth topological product". Call a model $(\mathfrak{A}, q_1, q_2, \ldots)$ *complete* if q_k is the kth topological product of q_1 on A.

We then obtain the following theorems whose proofs will appear in Sgro [11].

Theorem 4.4.1. *Let T be an $L(Q^n_{n\in\omega})$ theory where f_α, $a \in I$, is an (α_1, α_2)-ary function symbol. Then there is an axiom schema (AA) such that there is a complete topological model of T where each f is continuous if and only if T is consistent with (AA).*

As a corollary we have obtained:

Corollary 4.4.2. *Let T be an $L(Q)$ theory. Then there is a topological model $(\mathfrak{A}, q)$ which is a topological group if and only if* T *is consistent with the basic* L(Q) *axioms, group axioms, and*

$$Qx\varphi(x) \to Qx\varphi(t),$$

where

$$t \in (y^{\varepsilon(1)}_{\sigma(1)} \cdot y^{\varepsilon(2)}_{\sigma(2)} \cdot \ldots \cdot y^{\varepsilon(k)}_{\sigma(k)}/x)$$

$$\sigma : k+1 \to k+1 \quad \text{and} \quad \varepsilon : k+1 \to \{1, -1\}.$$

Theorem 4.4.3. *Let T be an $L(Q^n_{n\in\omega})$ theory which is consistent with (AA) and $Q^2xy\ (x \neq y)$,* (i.e. The Hausdorff Separation Axiom), *then T has a complete topological model which is also 0-dimensional and normal (paracompact).*

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